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Development of an Improved Two-Dimensional Finite-Element Code for Cylindrically Symmetric Eigenmodes

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***Abstract*—A new two-dimensional finite-element (FE) eigenmode solver has been developed, which is suitable for calculating cylindrically symmetric modes. The quantity H_θ/r is used in the code to describe the electromagnetic fields instead of H_θ or rH_θ, which is preferentially used in the existing codes, and the new formulation with H_θ/r is found to show higher accuracy and smoother convergence with respect to the number of mesh points. Comparison is also made between linear and quadratic elements, resulting in remarkably higher accuracy by the latter.**

***Index Terms*—Cavity eigenmode, finite-element method.**

I. Introduction

Cylindrically symmetric cavities are utilized in many radio-frequency (RF) devices, such as klystrons, RF guns, and various accelerating structures in particle accelerators. Many computer codes [1]–[11] have been developed thus far, and are in use for RF cavity designing for more than 30 years.

For cylindrically symmetric standing-wave modes, probably the most commonly used code would be the SUPERFISH [2], which calculates eigenfrequencies and corresponding angular magnetic field H_θ at the mesh points using the finite-differential method (FDM) with triangular meshes. However, depending on the cavity geometry, it is sometimes not accurate enough or, in other words, takes too much central processing unit (CPU) time and computer memory to achieve required accuracy. Since both higher accuracy and less computational efforts are always important from the viewpoint of saving time and effort for the users, continuous improvements of greater extent are called for in specific problems.

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From this viewpoint, a new two-dimensional-code Kyoto University eigenmode solver (KUEMS) has been developed, which is aimed at improved calculations of cylindrically symmetric $TM_{0\,nm}$ modes applicable to klystron simulations [12]. Instead of H_θ or rH_θ preferentially used in the existing codes [1]–[3], [7], the KUEMS uses: the quantity H_θ/r to describe the electromagnetic fields, which has the advantage of not requiring any boundary conditions on the symmetry axis, and the finite-element method (FEM) with quadratic triangular elements, which has high capability to model arbitrary structures.

This paper describes the numerical methods used in the KUEMS, followed by comparisons of the numerical results among the three different formulations, i.e., with H_θ/r, H_θ, and rH_θ, to show the advantageous features of this new formulation with H_θ/r with respect to accuracy in the eigenfrequencies, and convergence of the electric field on the symmetry axis. Comparisons between the linear and quadratic elements are also made to examine the accuracy together with the SUPERFISH.

II. Numerical Methods in the KUEMS

The numerical methods used in the KUEMS are described in this section, including the new FE formulation with the quantity H_θ/r. The essential difference from the other formulations with H_θ and rH_θ is described in Section II-B.

A. Basic Equations and Weak Formulation

For resonant electromagnetic fields, we can assume electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ at a time t and location $\boldsymbol{r}$ by

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{e}(\boldsymbol{r})\ \mathrm{Re}[\exp(i\omega t)], \tag{1a}$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \sqrt{\epsilon_0/\mu_0}\ \boldsymbol{h}(\boldsymbol{r}\ \mathrm{Re}[i\exp(i\omega t)] \tag{1b}$$

where ϵ_0 and μ_0 are the permittivity and permeability in free space, $f = \omega/2\pi$ is the resonant frequency, and $\boldsymbol{e}$ and $\boldsymbol{h}$ are the eigenmode patterns. Then, from Maxwell's equations in free space, the frequency and the magnetic field are expressed by the following eigenvalue problem:

$$\nabla \times \nabla \times \boldsymbol{h} = k^2 \boldsymbol{h} \text{ in } \Omega \tag{2}$$

$$\boldsymbol{n} \times (\nabla \times \boldsymbol{h}) = \boldsymbol{0} \text{ on } \Gamma \tag{3}$$

and the corresponding electric field is given by

$$\boldsymbol{e} = \frac{1}{k}\ \nabla \times \boldsymbol{h} \text{ in } \Omega \tag{4}$$

where $k = \omega/c$, the boundary Γ, and the domain Ω are the inner surface of the cavity wall and its volume, respectively, and $\boldsymbol{n}$ is the unit vector normal to Γ.

Satisfaction of both (2) and (3) is equivalent to the following formulation:

$$\int_\Omega \boldsymbol{v} \cdot [k^2\boldsymbol{h} - \nabla \times (\nabla \times \boldsymbol{h})]\ dv + \int_\Gamma \boldsymbol{v} \cdot [\boldsymbol{n} \times (\nabla \times \boldsymbol{h})]\ dS = 0, \quad \text{for any } \boldsymbol{v} \tag{5}$$

where dV and dS are a volume element in Ω and a surface element on Γ, respectively. With Gauss' divergence theorem, (5) can be reduced to the following well-known weak formulation [8]:

$$\int_\Omega (\nabla \times \boldsymbol{v}) \cdot (\nabla \times \boldsymbol{h})\ dV = k^2 \int_\Omega \boldsymbol{v} \cdot \boldsymbol{h}\ dV, \quad \text{for any } \boldsymbol{v}. \tag{6}$$

B. Choice of the Independent Variable

In the KUEMS, the quantity h_θ/r is chosen as the independent variable instead of h_θ or rh_θ preferentially used in the existing codes, exclusively due to the fact that no special treatment of the symmetry axis ($r = 0$) is required as follows.

With $\phi = h_\theta/r$ and $u = v_\theta/r$, (6) can be expressed by

$$\int_\Sigma \left[r^3 \left(\frac{\partial u}{\partial z}\frac{\partial \phi}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial \phi}{\partial r} \right) + 2r^3 \left(u\frac{\partial \phi}{\partial r} + \frac{\partial u}{\partial r}\phi \right) + 4ru\phi \right] dS = k^2 \int_\Sigma r^3 u\phi \, dS, \quad \text{for any } u \tag{7}$$

where Σ is the cross section of the domain Ω in z–r plane (hereafter, this will be referred to as "H/r-formulation"). In contrast, choice of $h = h_\theta, v = v_\theta$ ("H-formulation") and choice of $\xi = rh_\theta, w = rv_\theta$ ("rH-formulation") lead, respectively, to the following weak formulations:

$$\int_\Sigma \left[r\left(\frac{\partial v}{\partial z}\frac{\partial h}{\partial z} + \frac{\partial v}{\partial r}\frac{\partial h}{\partial r} \right) + \left(v\frac{\partial h}{\partial r} + \frac{\partial v}{\partial r}h \right) + \frac{vh}{r} \right] dS = k^2 \int_\Sigma rvh \, dS, \quad \text{for any } v \tag{8}$$

$$\int_\Sigma \frac{1}{r}\left(\frac{\partial w}{\partial z}\frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial r}\frac{\partial \xi}{\partial r} \right) dS = k^2 \int_\Sigma \frac{w\xi}{r} \, dS, \quad \text{for any } w. \tag{9}$$

It is clearly seen that, for the integration of (8) with the term vh/r on the left-hand side, $h = 0$ is always required on the axis to avoid infinity, and also for (9), $\nabla\xi = 0$ is always additionally required on the axis, both of which are consequently equivalent to the cylindrically symmetric conditions $h_\theta = e_r = 0$ and the existence of finite e_z on the axis. On the other hand, in this H/r-formulation, these requirements are automatically satisfied so long as ϕ and $\nabla\phi$ are finite, since h_θ, e_z, and e_r are all given in terms of ϕ by

$$h_\theta = r\phi \quad ke_z = 2\phi + r\frac{\partial \phi}{\partial r} \quad ke_r = -r\frac{\partial \phi}{\partial z}. \tag{10}$$

Thus, the H/r-formulation is shown not to require any special treatments on the symmetry axis, and, with the following additional reasons, it is applied to the KUEMS as follows.

1) Analytical integration of (7) is easily carried out in the FE formulation (described in Section II-C), while for the other formulations, special treatments are required on the axis;
2) H/r-formulation is expected to result in higher accuracy in the fundamental TM_{010} modes since $h_\theta/r \approx$ constant near the axis;
3) H/r-formulation is found to result in smoother convergence in calculating the electric field on the axis, as will be shown in the Section III.

C. FE Formulation

To numerically solve the weak formulation of (7), an N-dimensional subspace is applied to both the unknown function ϕ and the test function u. Suppose ϕ is a linear combination of N basis functions $\{\psi_i\}$, and an infinite number of test functions u are reduced to N test functions. Then, with

$$\phi(z,r) = \sum_{j=1}^{N} x_j \psi_j(z,r) \quad u(z,r) = \psi_i(z,r) \tag{11}$$

the weak formulation of (7) can be reduced to a Galerkin formulation

$$\sum_{j=1}^{N} a_{ij}x_j = k^2 \sum_{j=1}^{N} b_{ij}x_j, \quad \text{for any } 1 \le i \le N \tag{12}$$

or an algebraic eigenvalue problem

$$\boldsymbol{Ax} = k^2\boldsymbol{Bx} \tag{13}$$

where

$$a_{ij} = 2\pi \int_\Sigma \left\{ r^3 \left(\frac{\partial \psi_i}{\partial z}\frac{\partial \psi_j}{\partial z} + \frac{\partial \psi_i}{\partial r}\frac{\partial \psi_j}{\partial r} \right) + 2r^2 \left(\psi_i \frac{\partial \psi_j}{\partial r} + \frac{\partial \psi_i}{\partial r}\psi_j \right) + 4r\psi_i\psi_j \right\} dS \tag{14}$$

$$b_{ij} = 2\pi \int_\Sigma r^3 \psi_i \psi_j \, dS. \tag{15}$$

The FE formulation is a Galerkin formulation with a particular set of basis functions $\{\psi_i\}$. The KUEMS uses well-known Lagrange-type quadratic basis functions; the domain Σ is divided into FE's Σ_θ with N nodes, and the basis function ψ_i is defined as:

1) $\psi_i = 1$ at the node i;
2) $\psi_i = 0$ at the others;
3) continuous in Σ;
4) piecewise smooth in Σ_θ.

III. Numerical Tests

To verify the accuracy of the developed KUEMS, calculations were carried out for analytically solvable modes in a test cavity. The numerical error from the ideal value is, in general, mainly contributed by: 1) the error due to inaccurate modeling of geometric shape with a finite number of elements and 2) the discretization error of the formulation. To restrict discussions here to the latter one, a pillbox, whose cross section in the z–r plane has no curvature, is chosen as the test cavity.

A. Comparison Among Three Different Formulations

To verify the accuracy of the H/r-formulation used in the KUEMS, calculations were performed using three different formulations, namely, the H/r-, H-, and rH-formulations with the linear elements. The structure used here is a pillbox of 1-m radius and length.

Fig. 1 shows the relative errors of eigenfrequencies f and cavity voltages V from the analytic solutions, where V is defined by

$$V = \int_{r=0} |e_z| \, dz. \tag{16}$$

As for the frequencies f, it is found that, as was expected, the H/r-formulation results in remarkably higher accuracy for the fundamental TM_{010} mode than the H- and rH-formulations, but with an almost similar accuracy for higher modes. However, high accuracy for the fundamental mode is quite desirable since it is most commonly utilized in klystrons and RF guns. This advantage results from the fact that $\phi = h_\theta/r \approx$ constant for the TM_{010} mode near the symmetry axis.

On the other hand, in the cavity voltages V, the H/r-formulation seems to show less accuracy for some modes, e.g., (TM_{020} mode in Fig. 1(c). However, it should be noted that the voltages calculated with the H/r-formulation are found to converge smoothly as N increases, while those with the H- and the rH-formulations do not. Smooth convergence is remarkably important since rough convergence would make extrapolation difficult. This advantage also results from the fact that the quantity $\phi = h_\theta/r$ used in the H/r-formulation has a higher degree of freedom around the axis (compared with the

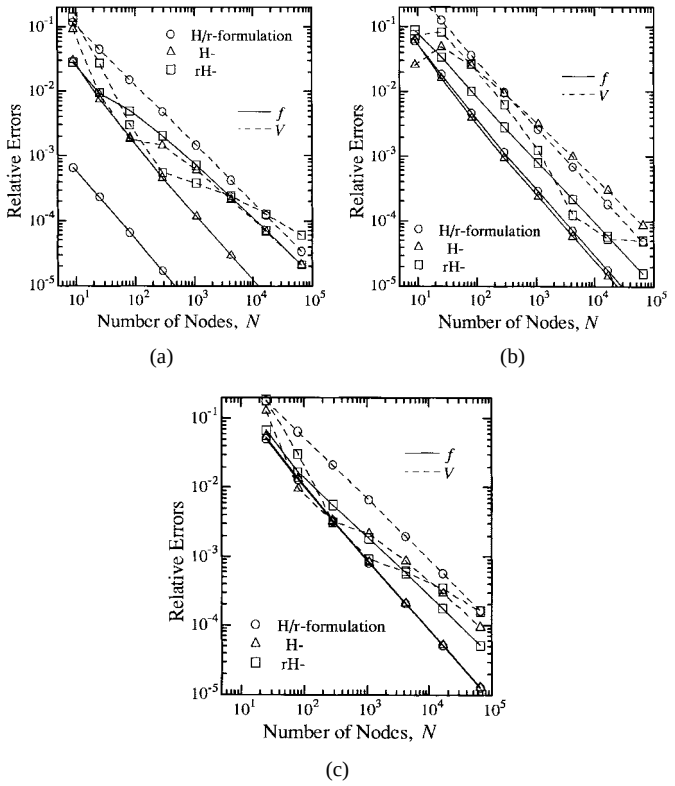

Fig. 1. Relative errors in eigenfrequencies and cavity voltages for (a) TM_{010}, (b) TM_{011}, and (c) TM_{020} modes in the pillbox cavity, comparing three different formulations.

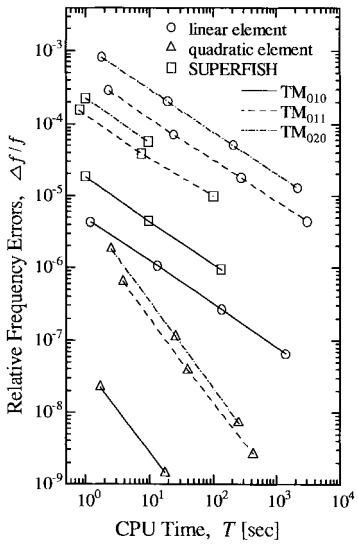

Fig. 2. Relative frequency errors as functions of CPU time (comparing the linear elements, quadratic elements, and SUPERFISH).

other formulations by $h = h_\theta$ or $\xi = rh_\theta$) because no condition is applied to the axis in the H/r-formulation.

B. Comparison Between Linear and Quadratic Elements

Since the quadratic elements are generally regarded as showing a higher accuracy than the linear elements for the same number of nodes, they were actually applied to some eigenmode solvers [3], [6]–[8] as well as to the present KUEMS. However, it is not obvious whether they may result in higher accuracy within the same CPU time, in which the users are very much interested. To make this clear, comparisons were made between the linear and quadratic elements with the H/r-formulation for the $\mathrm{TM}_{0\mathrm{nm}}$ modes in the pillbox. For comparison, the SUPERFISH, which uses the first-order FDM, is also applied to the same modes.

The relative frequency errors are found to scale as $\Delta f/f \propto N^{-2.0\sim-2.2}$ for the quadratic elements, while $\Delta f/f \propto N^{-1.0\sim-1.1}$ for the linear elements and the SUPERFISH. It is also important that the CPU time T is found to increase with the same order for either the linear or quadratic elements as N increases. Consequently, as shown in Fig. 2, the quadratic element scheme is found to take the least CPU time among the three for all the three modes.

IV. Summary

A new two-dimensional code has been developed based on the FEM, making it ideal for calculating cylindrically symmetric eigenmodes. The quantity H_θ/r, which has the advantage of not requiring any conditions on the symmetry axis, is used to represent the electromagnetic fields instead of H_θ or rH_θ, which have been used thus far in the existing codes.

It is found that the new FE formulation with H_θ/r results in remarkably higher accuracy in the eigenfrequency of the fundamental mode, but no less accuracy in the other higher modes. It also results in smoother convergence of the calculation of the electric field on the symmetry axis with respect to number of the mesh points.

It is also found that by use of the quadratic elements, faster convergence can be achieved compared with the linear elements or the SUPERFISH.

All these results are remarkable from the viewpoint of saving time and effort for both computers and their users.

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